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APPLICATION OF MULTI BLOCK METHOD FOR SOLUTION OF TWO-DIMENSIONAL TRANSIENT INVERSE HEAT CONDUCTION PROBLEMS

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Abstract - In this study a multi block procedure is implemented to solve accurately two-dimensional transient Inverse Heat Conduction Problems (IHCPs). The multi block method is implemented for geometric decomposition of physical domain into regions with blocked-interface structured grids. The Finite Element Method (FEM) with Galerkin weighting function is employed for direct solution of transient heat conduction equation. Inverse algorithms used in this research are iterative Levenberg-Marquardt and adjoint conjugate gradient techniques for parameter estimation. To have fewer numbers of unknown coefficients for estimation, polynomials are used for parameterization of the source term. The measured transient temperature data needed in the inverse solution are given by exact or noisy data. Estimations of unknown time varying strength of heat sources are obtained for the solution of two-dimensional transient IHCPs and the results of the present study for coefficients of unknown heat source functions are compared to those of exact heat sources.

1. INTRODUCTION

Recently, new methods are introduced to promise improved understanding and modeling of physical processes. In heat transfer research, two ideas are commonly considered to solve heat transfer problems. These ideas are the use of the direct analytical or numerical schemes as the traditional approach and the use of the inverse analysis techniques as recently used idea.

Mathematically, IHCPs, unlike the direct heat conduction problems, belong to a class of “ill-posed” problems which do not satisfy the “well-posed” conditions introduced by Hadamard, [1]. The IHCPs are very sensitive to random errors in the measured temperature data, thus special techniques are needed for their solutions in order to satisfy the stability condition, or so called “well-posed” conditions. Minimization of error is the main objective of the inverse analysis which is related to analytical design theory. A number of parameter and function-estimation schemes for inverse analysis have been proposed to treat the ill-posed nature of IHCPs, [2-3].

In the direct problem of the IHCP, for composite structures, single structured grid is not suitable in many realistic applications and composite grid methods must be used. A composite grid is a union of structured grids, unstructured grids or both of them. In this method, the physical domain is divided into subdomains in which each subdomain is discretized into a subgrid with two layers of points or cells overlapped into neighboring subdomains. Within each subregion regardless of the shape or complexity of the physical region, the different subgrids interact with each other via block interface with the periodic exchange of boundary information between neighboring subdomains. In addition, in each subdomain, the direct governing equation and other Partial Differential Equations (PDEs) resulting from the inverse analysis, are resolved in an independent manner. This permits the solution of large problems requiring many mesh points by keeping only the information needed to solve governing equations in one block, in the RAM of computer while storing the information of remaining blocks in the hard disk, [4].

This study is motivated by the use of multi block method employing structured grids capable of providing accurate solutions of the IHCPs in industrial configurations, including composite structures. IHCP solver with structured multi-block grids can be a suitable computational fluid dynamics technique for studying practical problems in heat transfer field. The use of multi block IHCP solvers is good to estimate unknown parameters and functions in composite structures.

Among numerical schemes, FEM is another method suitable for numerical approximation of the PDEs. To have flexibility and easy communication between two neighboring blocks, the advantages of FEM in boundary treatment is used in this study as the numerical solution routine for computations of direct heat conduction problem. In addition, the frontal technique of equation assembly and reduction is used. The frontal solution is a very efficient direct solution process and it is designed to minimize core storage requirements, the number of arithmetic operations, and the use of peripheral equipment, [5].

A variety of numerical and experimental studies for estimation of unknown heat sources in two-dimensional steady/transient IHCPs have been presented in the literature over the last two decades. Some studies on point/line heat sources estimation have been proposed in [6-20]. In these works, different inverse analysis algorithms and numerical schemes such as finite element, boundary element and finite difference methods have been used. The proposed methods have been applied to the strength estimation of one or two heat sources in 1D and 2D cases when the location of the sources is known a priori.

In this study, application of the multi block method for solution of two-dimensional transient IHCPs is presented. The physical domain is decomposed into subdomains with blocked-interface grids. A structured grid is generated by algebraic grid generation method for each subdomain independent of other blocks. The numerical scheme for solution of transient heat conduction equation is the FEM together with the frontal technique to solve algebraic system of discrete equations. Inverse algorithms used in this study are parameter estimation of the iterative Levenberg-Marquardt and adjoint conjugate gradient algorithms. To have fewer numbers of unknown coefficients for estimation, polynomials are used. The measured transient temperature data needed in the inverse solution are given by exact or inexact (noisy) data. To show the performance of the multi block method, estimations of unknown time varying strength of heat sources are obtained for the solution of two-dimensional transient IHCPs. The results of the present study for coefficients of unknown heat source functions are compared to those of exact heat sources.

2. MULTI BLOCK METHOD

An engineering code requires geometrical flexibility and high accuracy. Additional requirements include the demand for reasonable turnaround times and high robustness. These are somewhat contradictory. Robust numerical schemes can possess relatively low accuracy as their robustness is usually achieved by inclusion of artificial dissipation into the numerical scheme, either explicitly or implicitly. Higher-order-accurate numerical schemes are usually far less robust, and their incorporation into multi block framework is usually inefficient. As a result, the practical use of high-accuracy schemes is restricted to relatively simple geometries.

In a multi block approach the global domain is divided into smaller blocks for which computational meshes are easier to generate. It is presumed that an iteration process with global time interval is applied globally to all of blocks, with data exchange among the blocks responsible for the validity of boundary conditions on the block interfaces and their immediate vicinity. However, numerical problems manifest themselves when we need to implement the boundary conditions and maintain the conservativity of numerical fluxes in the vicinity of block interfaces.

For IHCP solver, more changes to the numerical approximation have been required in order to simplify the interface treatment in the case of geometrical domain decomposition. But, this leads to only insignificant loss of accuracy in the neighborhood of block interfaces and has almost no effect on the overall numerical results.
In this study an attempt is made to challenge the goal of combining a multi block technique, geometrically flexible and well suited for parallel computations, with a low-dissipation numerical method, thus ensuring robustness and efficiency of the resulting method. This work is based on the previous publications related to a multi block solver, [4] and a single-block IHCP solver, [21]. Cell-to-cell matching (blocked grids) across the interface of neighboring blocks is used, thus, the communication overhead caused by the data exchange among neighboring blocks is negligible.

3. GOVERNING EQUATION

The governing equation in dimensionless form is the two-dimensional transient heat conduction equation with heat source. This equation expressed in Cartesian coordinate system is written:

$$\frac{1}{\alpha}\frac{\partial T}{\partial t}=\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+G(x,y,t)\ , \qquad G(x,y,t)=g(t)\delta(x-x_s,y-y_s) \qquad t>0\,,\ 0<x<1\,,\ 0<y<1 \qquad (1)$$

where function $g(t)$ is time varying strength of the source located at $(x_s,y_s)\in(0,1)\times(0,1)$, δ is the Dirac delta function and α is the dimensionless thermal diffusivity. Also, x_s and y_s show the source location. The initial and boundary conditions are as follows:

$$\begin{aligned} &T(x,y,t)=0 \qquad && x=0,1,\quad 0<y<1,\quad t>0 \\ &T(x,y,t)=0 && 0<x<1,\quad y=0,1,\quad t>0 \\ &T(x,y,t)=0 && 0<x<1,\ 0<y<1,\ t=0 \end{aligned} \qquad (2)$$

The dimensionless groups are as follows:

$$t=\frac{t^*}{\alpha_{ref}\,/\,L_{ref}^2}, \qquad x=\frac{x^*}{L_{ref}}, \qquad y=\frac{y^*}{L_{ref}}, \qquad T=\frac{T^*}{T_{ref}}, \qquad \alpha=\frac{\alpha^*}{\alpha_{ref}}, \qquad G=\frac{G^*}{k\,T_{ref}\,/\,L_{ref}^2} \qquad (3)$$

where k is thermal conductivity and L_{ref} , T_{ref} and α_{ref} are reference length, temperature and thermal diffusivity, respectively. It is assumed that these parameters are constant and known.

4. NUMERICAL SCHEME

Each IHCP algorithm, regardless of its theoretical approach, requires the use of a suitable numerical solution routine for direct heat conduction and other PDEs such as sensitivity equation. This solution routine may be called upon numerous times by the main IHCP computational routine.

In this paper, a FEM is used for direct solution. The Galerkin form of the weighted residuals procedure is used to formulate the FEM. Therefore, eqn.(1) multiplied by weighting function, integrated by part and employing Gauss theorem over the domain Ω leads to, [5]:

$$\int_{\Omega}\left(\frac{w}{\alpha}\frac{\partial T}{\partial t}+\frac{\partial w}{\partial x}\frac{\partial T}{\partial x}+\frac{\partial w}{\partial y}\frac{\partial T}{\partial y}\right)d\Omega=\int_{\Omega} w\,G(x,y,t)d\Omega+\int_{\Gamma} w\left[\frac{\partial T}{\partial x}n_x+\frac{\partial T}{\partial y}n_y\right]d\Gamma \quad (4)$$

In the above equation, w is weighting function and Γ is boundary of the domain Ω. In addition, n_x and n_y are the components or direction cosines of the unit outward vector normal to the boundary. A four nodes element is used, along with bilinear interpolation to approximate temperature. The Galerkin weighting function and temperature in the four nodes element are defined by:

$$w_i=N_i, \qquad T^e=\sum_{j=1}^{4}N_jT_j \quad (5)$$

where N_i is the shape function of each node in each element. By using the above relations in eqn.(4), the final matrix form of this equation after linearization of the time-dependent derivative of temperature

$$\frac{\mathbf{T}^{n+1}-\mathbf{T}^{n}}{\Delta t}=(1-\theta)\dot{\mathbf{T}}^{n}+\theta\dot{\mathbf{T}}^{n+1} \quad (6)$$

is

$$\hat{\mathbf{K}}^{n+1}\mathbf{T}^{n+1}=\hat{\mathbf{K}}^{n}\mathbf{T}^{n}+\hat{\mathbf{F}}^{n} \quad (7)$$

where matrices $\hat{\mathbf{M}}$ and $\hat{\mathbf{K}}$ and vector $\hat{\mathbf{F}}$ are determined as follows:

$$\hat{\mathbf{K}}^{n+1}=\mathbf{M}^{n}+\theta\,\Delta t\,\mathbf{K}^{n}, \qquad m_{ij}=\frac{1}{\alpha}\int_{\Omega}N_iN_jd\Omega$$

$$\hat{\mathbf{K}}^{n}=\mathbf{M}^{n}-(1-\theta)\Delta t\,\mathbf{K}^{n}, \qquad k_{ij}=\int_{\Omega}\left[\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}+\frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}\right]d\Omega \quad (8)$$

$$\hat{\mathbf{F}}^{n}=\left[(1-\theta)\mathbf{F}^{n}+\theta\mathbf{F}^{n+1}\right]\Delta t, \qquad f_i=\int_{\Omega}N_i\,G(x,y,t)\,d\Omega+\int_{\Gamma}N_i\left(\frac{\partial T}{\partial x}n_x+\frac{\partial T}{\partial y}n_y\right)d\Gamma$$

where θ is a constant and can be assigned 0.0, 0.5 and 1.0 corresponding to explicit, Crank-Nicolson and implicit schemes, respectively. Frontal technique based on Gaussian reduction process is used to solve eqn.(7). This solution requires fewer calculations and less computer storage space than banded solution since space is only allocated when required by non-zero row coefficients in the reduction process. Also, in each block, grids have been generated by an algebraic method, [22].

5. INVERSE ANALYSIS SCHEME

In inverse analysis, unlike direct solution, there are one or more parameters or functions which must be determined from the sensor values of temperature measured inside the field or on the boundaries, [14, 23]:

$$[\mathbf{T}^{m}]^{T}=\left[T_1^{m},\,T_2^{m},\ldots\ldots,\,T_M^{m}\right] \quad (9)$$

where M is the total number of sensors. In inverse analysis, error is defined by the difference between measured (T^m) and computed (T^c) temperatures:

$$\mathbf{e}=\mathbf{T}^{m}-\mathbf{T}^{c} \quad (11)$$

Minimization of this error is the main goal of inverse analysis. The synthetic experimental data is generated by adding random noise to the exact calculated values of the temperatures

$$\mathbf{T}^{m}=\mathbf{T}^{exact}+\boldsymbol{\varepsilon} \quad (12)$$

where $\boldsymbol{\varepsilon}$ is the error due to measuring instruments. One of the minimization strategies for eqn.(11) is to apply the least squares method (I is the total number of time measurements at each sensor location)

$$S(\mathbf{P})=(\mathbf{T}^{m}-\mathbf{T}^{c})^{T}(\mathbf{T}^{m}-\mathbf{T}^{c}) \qquad \text{or} \qquad S(\mathbf{P})=\sum_{j=1}^{M}\sum_{i=1}^{I}\left(T_{ij}^{m}-T_{ij}^{c}(\mathbf{P})\right)^2 \quad (13)$$

$\mathbf{P}$ is the vector of unknown parameters. Another minimization strategy is the weighted least squares method:

$$S(\mathbf{P})=(\mathbf{T}^{m}-\mathbf{T}^{c})^{T}\,\mathbf{W}(\mathbf{T}^{m}-\mathbf{T}^{c}) \quad (14)$$

Note that inverse problems are sensitive to measurement errors in input data due to location of sensors and frequency of oscillations, and the solution of these problems is affected by such errors which play a crucial role regarding the stability and accuracy of the solution. Thus, standard assumptions as proposed by Beck, [2-3] should be applied to measurement errors in input data. In this study, zero mean, unitary standard deviation, and normal distribution for the measurement errors have been used for solution of inverse problem, [14, 23].

For parameter estimation, a lot of powerful methods have been worked out so far. Beck, [2] has described one possible approach based on the sensitivity coefficient concept. The sensitivity coefficient is quite instrumental and plays an important role in parameter estimation approach. In fact, the sensitivity coefficient is a measure of the sensitivity of the estimated temperature with respect to changes in the parameters, $\mathbf{J} = \partial \mathbf{T} / \partial \mathbf{P}$.

It can be easily noticed that the estimation of the parameters is extremely difficult for small values of sensitivity coefficients. In such a case, basically the same value for temperature would be obtained for a wide range of values of parameters and the inverse problem is ill-posed. Thus, it is desirable to have linearly-independent sensitivity coefficients with large magnitude, so that the inverse problem is not very sensitive to measurement errors and accurate estimates of the parameters can be obtained. To calculate sensitivity coefficients, three different approaches can be considered. Since sensitivity coefficient is the derivative of temperature with respect to the unknown quantity, [23]:

1. It can be represented by a fraction of two finite differences, $\mathbf{J} = \Delta \mathbf{T} / \Delta \mathbf{P}$.
2. It can be determined by differentiating the original direct problem with respect to the unknown coefficients, [24-25].
3. It is determined by differentiating the analytical solution with respect to the unknown coefficients, if the direct problem is linear and an analytical solution is available for the temperature field.

For linear inverse problems, the sensitivity coefficients are computed only once, starting with initial and boundary conditions for some sequential steps or a whole step, whereas, for nonlinear problems, the sensitivity coefficients should be recalculated for each time/spatial interval whenever the temperature field is updated.

5.1. INVERSE ANALYSIS ALGORITHMS

For the minimization of the least squares norm eqn.(13), two iterative algorithms based on the sensitivity coefficient are considered, the Levenberg-Marquardt and the adjoint conjugate gradient methods for parameter estimation, [23]. Solution of the inverse problems using these algorithms involves some sequential items, such as solution of the direct problem, solution of the inverse problem, a stopping criterion and iterative process. These methods can be suitably arranged in iterative procedures of the form:

$$\mathbf{P}^{k+1} = \mathbf{P}^{k} + \Delta \mathbf{P}^{k} \tag{15}$$

It is noted that in this study there is a priori information of the functional form of the heat sources strengths and their locations (see Section 6.1., eqn.(35), and Section 6.2., eqn.(36), for details). Also, we have assumed that the number of these functions and their coefficients are known. But, for cases with no a priori information of the functional forms, one must use the function-estimation methods of the inverse analysis, such as adjoint conjugate gradient method, [14, 23] and function specification method, [26-27]. The solution of the function estimation problems is highly sensitive to noise in the measured data. Even if data could be measured exactly, the inverse function estimation problems would be difficult.

5.1.1. THE LEVENBERG-MARQUARDT ALGORITHM

The Levenberg-Marquardt method has been applied to the solution of a wide range of inverse problems for estimation of unknown parameters. In this algorithm, after solving the direct problem, the inverse problem must be solved. To solve this problem, it is assumed that unknown heat source strength, $g(t)$, is parameterized in the form of a polynomial function with N coefficients. N is the total number of unknown parameters (coefficients) and is known:

$$g(t) = \sum_{j=1}^{N} P_j \, C_j(t) \tag{16}$$

In above equation, $C_j(t)$ for each heat source are known time dependent trial functions. Thus, the problem of the function estimation is converted to the problem of the parameter estimation. For minimization of eqn.(13), the gradient of $S(\mathbf{P})$ with respect to the vector of the unknown parameters must be equated to zero:

$$\nabla S(\mathbf{P}) = -2\,\mathbf{J}^{T}(\mathbf{P})\,[\mathbf{T}^{m} - \mathbf{T}^{c}(\mathbf{P})] = 0 \tag{17}$$

To solve the above equation, the sensitivity matrix $\mathbf{J}$ needs to be calculated. This matrix is determined by differentiating eqn.(1) with respect to the vector of the unknown coefficients [24-25]:

$$\frac{1}{\alpha}\frac{\partial \mathbf{J}}{\partial t} = \frac{\partial^2 \mathbf{J}}{\partial x^2} + \frac{\partial^2 \mathbf{J}}{\partial y^2} + \mathbf{C}(t)\,\delta(x - x_s, y - y_s) \qquad t > 0\,,\ 0 < x < 1\,,\ 0 < y < 1 \tag{18}$$

Note that for the initial condition and all the boundary conditions except the block-interface boundary, the sensitivity matrix $\mathbf{J}(x, y, t)$ is equal to zero.

For nonlinear inverse problems the sensitivity coefficients depend on the unknown parameters, so one needs to use a Taylor series expansion around the current solution $\mathbf{P}^k$. Therefore, we can determine the vector of parameters with the following equation using Gauss Method:

$$\mathbf{P}^{k+1} = \mathbf{P}^k + \left[(\mathbf{J}^k)^T \mathbf{J}^k\right]^{-1} (\mathbf{J}^k)^T \left[\mathbf{T}^m - \mathbf{T}^c(\mathbf{P}^k)\right] \tag{19}$$

where k is the current iteration number. The implementation of the iterative procedure requires that the matrix $(\mathbf{J}^k)^T \mathbf{J}^k$ (Identifiability Condition) is non-singular (or its determinant is not equal to zero or not very small). To prevent this situation, Levenberg-Marquardt proposed the form of the iterative procedure, eqn.(19), on the following form:

$$\mathbf{P}^{k+1} = \mathbf{P}^k + \left[(\mathbf{J}^k)^T \mathbf{J}^k + \mu^k \Psi^k\right]^{-1} (\mathbf{J}^k)^T \left[\mathbf{T}^m - \mathbf{T}^c(\mathbf{P}^k)\right] \tag{20}$$

where μ is a damping parameter which must be a positive value and Ψ is a diagonal matrix which is defined as follows:

$$\Psi = \mathrm{Diag}\left[(\mathbf{J})^T \mathbf{J}\right] \tag{21}$$

The goal of the proposed form of eqn.(20) is to damp oscillations and instabilities due to the ill-posed character of the problem.

5.1.2. ADJOINT CONJUGATE GRADIENT ALGORITHM

This algorithm is a simple, straightforward and powerful iterative parameter-estimation algorithm for the solution of linear and nonlinear inverse problems using conjugate gradient method with adjoint problem. This method minimizes the objective function, eqn.(13), by applying a suitable search step size along the direction of descent in each step. In this algorithm, after solving the direct problem, to solve the inverse problem, the following relations for determination of the unknown parameter vector of heat source are considered, [23]:

$$\mathbf{P}_i^{k+1} = \mathbf{P}_i^k - \beta_i^k \mathbf{d}_i^k \qquad \text{where} \qquad \mathbf{d}_i^k = \nabla S(\mathbf{P}_i^k) - \gamma_i^k \mathbf{d}_i^{k-1} \tag{22}$$

where β_i^k, $\mathbf{d}_i^k$, and γ_i^k are the search step size, the direction of descent and the conjugate coefficient, respectively. $\nabla S(\mathbf{P}_i^k)$ is the gradient direction and subscript i is an index for representing each heat source function. In fact, in this method, the sensitivity and the adjoint problems are solved to determine the search step size and to obtain the gradient direction. It is important to note that there is only one sensitivity equation in this method, unlike the Levenberg-Marquardt method in which the number of the sensitivity equations depends on the number of estimation parameters.

For the sensitivity problem, the temperature distribution and the heat source functions are perturbed:

$$T(x,y,t) = \overline{T}(x,y,t) + \Delta T(x,y,t) \qquad \text{and} \qquad g_i(t) = \overline{g}_i(t) + \Delta g_i(t) \tag{23}$$

By using eqn.(23) in eqns (1) and (2), we can derive the following equation to solve the sensitivity problem and finally determine the derivative direction of the temperature, $\Delta T(x,t)$:

$$\frac{1}{\alpha}\frac{\partial \Delta T}{\partial t} = \frac{\partial^2 \Delta T}{\partial x^2} + \frac{\partial^2 \Delta T}{\partial y^2} + \Delta g(t)\,\delta(x - x_s, y - y_s) \qquad t > 0,\ 0 < x < 1,\ 0 < y < 1 \tag{24}$$

To solve the above equation, the sensitivity variable $\Delta T(x,y,t)$ is equal to zero for the initial condition and all the boundary conditions except the block-interface boundary.

In the adjoint problem, a Lagrange multiplier, $\lambda(x,y,t)$, is used in the objective function in order to satisfy a constraint for the temperature field. Therefore, the objective function can be rewritten in the following form:

$$S(\mathbf{P}) = \sum_{j=1}^{M} \int_{t=0}^{t_f} \left(T_j^m - T^c(x_j,y_j,t,\mathbf{P})\right)^2 dt + \sum_{j=1}^{M} \int_{t=0}^{t_f} \int_{x=0}^{1} \int_{y=0}^{1} \lambda(x,y,t)\left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + G(x,y,t) - \frac{1}{\alpha}\frac{\partial T}{\partial t}\right] dt\,dx\,dy \tag{25}$$

To calculate the Lagrange multiplier, the temperature, the heat source functions and the objective function $S(\mathbf{P})$ are perturbed in eqn.(25) and after using some algebraic operations and integration by parts, we can write the following equation:

$$-\frac{1}{\alpha}\frac{\partial \lambda}{\partial t} = \frac{\partial^2 \lambda}{\partial x^2} + \frac{\partial^2 \lambda}{\partial y^2} + 2\sum_{j=1}^{M}\left(T^c(x_j,y_j,t,\mathbf{P}) - T_j^m\right)\delta\left(x - x_j, y - y_j\right) \quad t > 0,\ 0 < x < 1,\ 0 < y < 1 \tag{26}$$

and the Lagrange multiplier $\lambda(x,y,t)$ is equal to zero for the initial condition and all the boundary conditions except the block-interface boundary. Finally, the gradient direction is given in the following form:

$$\left[\nabla S(\mathbf{P}_i)\right]_j = \sum_{m=1}^{M} \int_{t=0}^{t_f} \lambda(x_s,y_s,t)\left[C_j(t)\right]_i dt \qquad \text{for} \quad j = 1,2,\ldots,N \tag{27}$$

One of the expressions for the conjugate coefficient is proposed by Fletcher-Reeves, [23]:

$$\gamma_i^k = \frac{\sum_{j=1}^{N} \left[\nabla S(\mathbf{P}_i^k) \right]_j^2}{\sum_{j=1}^{N} \left[\nabla S(\mathbf{P}_i^{k-1}) \right]_j^2} \qquad \text{for} \quad k = 1, 2, \ldots \tag{28}$$

To evaluate the search step sizes of the heat sources, we write the objective function for iteration $k+1$ as:

$$S(\mathbf{P}_1^{k+1}, \mathbf{P}_2^{k+1}, \ldots, \mathbf{P}_i^{k+1}) = \sum_{j=1}^{M} \int_{t=0}^{t_f} \left(T_j^m - T^c(x_j, y_j, t, \mathbf{P}_1^k - \beta_1^k d_1^k, \mathbf{P}_2^k - \beta_2^k d_2^k, \ldots, \mathbf{P}_i^k - \beta_i^k d_i^k) \right)^2 dt \tag{29}$$

Then, in the above equation, $T^c(x_j, y_j, t, \mathbf{P}_1^k - \beta_1^k d_1^k, \mathbf{P}_2^k - \beta_2^k d_2^k, \ldots, \mathbf{P}_i^k - \beta_i^k d_i^k)$ is linearized using a Taylor series expansion. Finally, the search step sizes is evaluated by minimizing the linearized objective function respect to β_i^k 's. For one heat source (see Section 6.1.), β^k can be obtained as follows:

$$\beta^k = \sum_{j=1}^{M} \int_{t=0}^{t_f} \Delta T_j(\mathbf{P}^k) \left[T_j^c(\mathbf{P}^k) - T_j^m \right] dt \Big/ \sum_{j=1}^{M} \int_{t=0}^{t_f} \left[\Delta T_j(\mathbf{P}^k) \right]^2 dt \tag{30}$$

and for two heat sources (see Section 6.2.), β_1^k and β_2^k can be obtained as follows:

$$\beta_1^k = \frac{F_1 A_{22} - F_2 A_{12}}{A_{11} A_{22} - A_{12}^2} \qquad \text{and} \qquad \beta_2^k = \frac{F_2 A_{11} - F_1 A_{12}}{A_{11} A_{22} - A_{12}^2} \tag{31}$$

where

$$A_{11} = \sum_{j=1}^{M} \int_{t=0}^{t_f} \left[\Delta T_j(\mathbf{P}_1^k) \right]^2 dt, \quad A_{22} = \sum_{j=1}^{M} \int_{t=0}^{t_f} \left[\Delta T_j(\mathbf{P}_2^k) \right]^2 dt, \quad A_{12} = \sum_{j=1}^{M} \int_{t=0}^{t_f} \left[\Delta T_j(\mathbf{P}_1^k) \right] \left[\Delta T_j(\mathbf{P}_2^k) \right] dt$$

$$F_1 = \sum_{j=1}^{M} \int_{t=0}^{t_f} \left[\Delta T_j(\mathbf{P}_1^k) \right] \left[T_j^c(\mathbf{P}_1^k, \mathbf{P}_2^k) - T_j^m \right] dt, \qquad F_2 = \sum_{j=1}^{M} \int_{t=0}^{t_f} \left[\Delta T_j(\mathbf{P}_2^k) \right] \left[T_j^c(\mathbf{P}_1^k, \mathbf{P}_2^k) - T_j^m \right] dt \tag{32}$$

5.2. STOPPING CRITERIA

The above two iterative algorithms need a criterion to stop the iterative procedure of solution. The stopping criterion, based on the Discrepancy Principle, [23] is:

$$S(\mathbf{P}^{k+1}) < \varepsilon \tag{33}$$

where ε is a user prescribed tolerance. The solution of inverse problems using iterative algorithm may become stable if the Discrepancy Principle is used to stop the iterative procedure, [23].

6. RESULTS

To show the performance of the multi block method to solve two dimensional IHCPs, solutions of two cases have been presented, namely, the estimation of a single heat source and the simultaneous estimation of two heat sources. In each case, both the Levenberg-Marquardt and the adjoint conjugate gradient algorithms have been use`d to estimate the parameters. The initial guesses of the strength of the sources are taken equal to zero. For the solution of the direct problem and other PDEs required in inverse analysis, the computational domain has been divided into two regions with separate algebraic grid generation, 16×8 and 16×8 grids, respectively. Figure 1

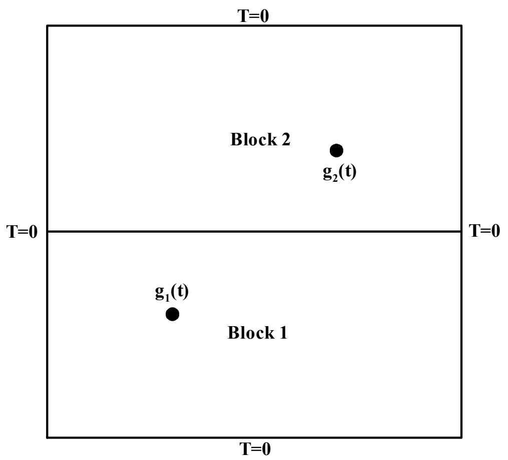


Figure 1. Schematic of the heat source positions and the boundary conditions of the problem.

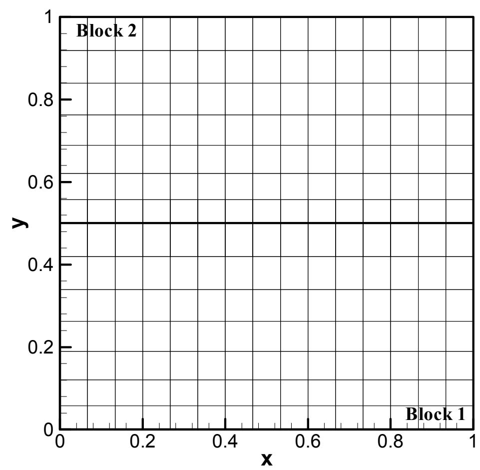


Figure 2. Grid generation in two dimensional decomposed field.

presents the blocks, the physical domain and the boundary conditions of the problem, and Figure 2 shows the grid used for the numerical discretisation of the PDEs involved.

The computational temperature field has been used to obtain measured temperature values of the sensors required for the solution of the inverse problem. Both exact and inexact measured data has been inverted. For the inexact values, normal distribution of the assumed noise with zero mean value has been added to the exact values of the computational field of the temperature, from $t = 0$ to $t = 3$:

$$T^m(x_m, y_m, t) = T^c(x_m, y_m, t) + \omega * 0.01 * T^c_{\max} \tag{34}$$

where ω is a normally distributed random number between -2.576 and 2.576 and the positions of the sensors are located near the upper boundary. The temporal increment is chosen as $\Delta t = 0.1$ and α is set equal to one.

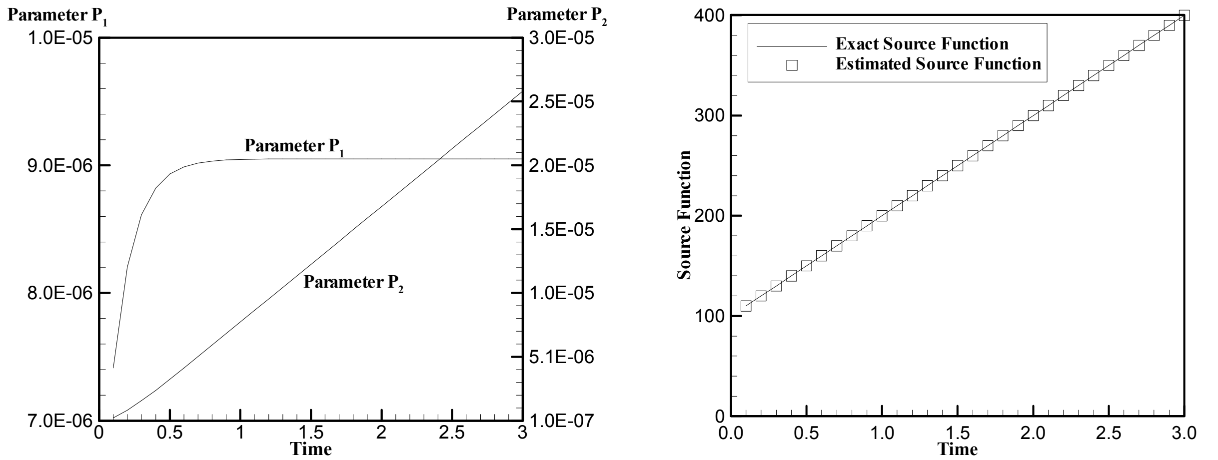
6.1. CASE 1

The purpose of this case is to estimate the unknown strength of a heat source depending on time, $g_1(t)$. $g_1(t)$ is a linear time varying function with constant parameters which has to be determined using the solution of the IHCP:

$$g_1(t) = P_1 + P_2 t = 100 + 100t \tag{35}$$

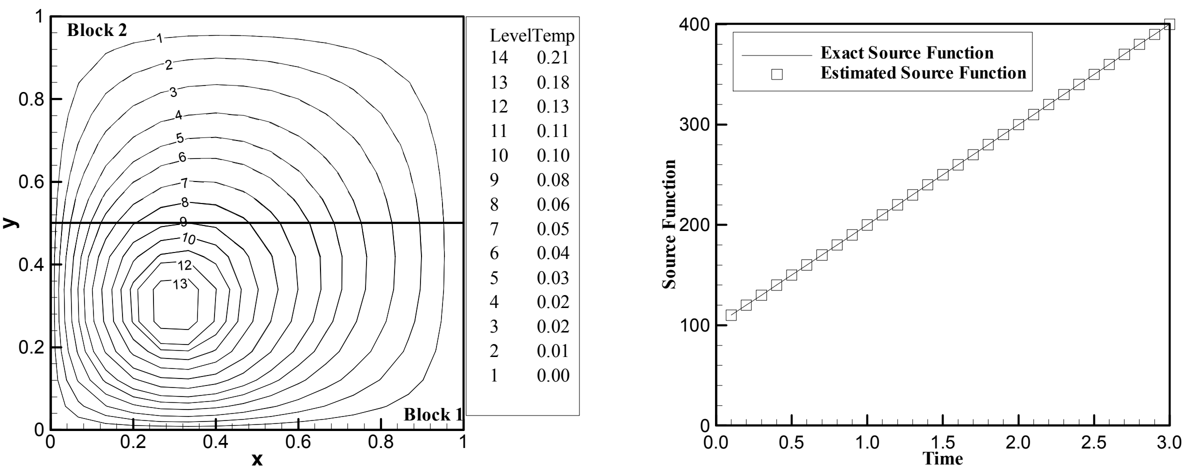
The source and the sensor are located at $x_s = 0.3$, $y_s = 0.3$ and at $x = 0.3$, $y = 0.95$, respectively. The graphical results of this case are presented in Figures 3 and 4 for the Levenberg-Marquardt and the adjoint conjugate gradient algorithms, respectively. The solution of this case is presented numerically in Table 1.

Figure 3.a shows the timewise variations of the sensitivity of the noisy measured temperature with respect to the parameters of function $g_1(t)$. It can be seen that the sensitivity of the sensor with respect to parameter P_1, at first, increases continuously versus time and after that, its value becomes constant. But the sensitivity of the sensor with respect to parameter P_2 goes up continuously versus time. Therefore, it can be verified that the conditions e.g. total time to measure temperature, for the estimation of these parameters are adequate. In Figure



a. Time variation of the sensor sensitivity to parameters. b. Variation of the heat source function versus time.

Figure 3. Solution of case 1 using Levenberg-Marquardt algorithm.



a. Temperature contours. b. Variation of the heat source function versus time.

Figure 4. Solution of case 1 using adjoint conjugate gradient algorithm.

3.b, the estimated function of the source strength has been compared with the exact function for the noisy measured temperature and shows good agreement.

Figure 4.a shows the temperature field computed by the direct problem of this IHCP with the heat source after estimating this function, at $t = 3$. Also, in Figure 4.b, the estimated function of the source strength has been compared with that of exact function for the noisy measured temperature and they verify each other.

Table 1: Estimation of a time-varying heat source in a two dimensional field.

Estimation Method	Function	Exact Measured Temperatures			Noisy Measured Temperatures		
		P_1	P_2	RMS* Error	P_1	P_2	RMS* Error
Levenberg-Marquardt	g_1	99.984	99.992	0.0306	99.087	100.52	0.5313
Adjoint Conjugate Gradient	g_1	99.600	100.30	0.3394	98.005	101.08	1.0953

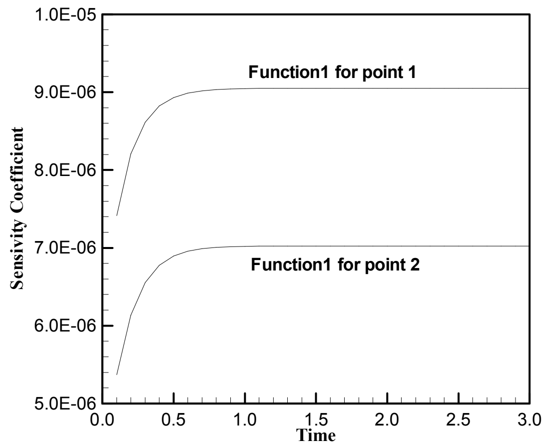
*Root Mean Square

6.2. CASE 2

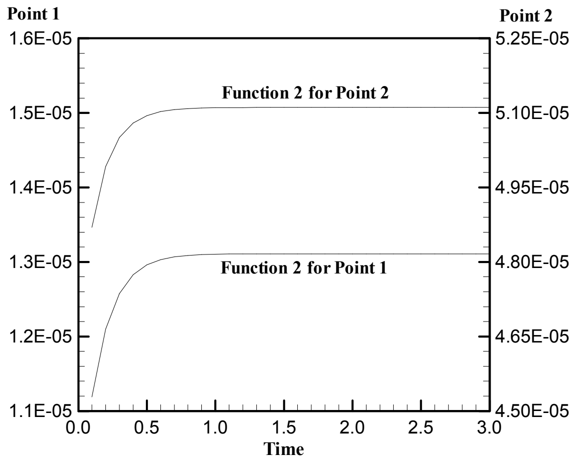
In this case, the simultaneous prediction of time-varying strengths of two heat sources, $g_1(t)$ and $g_2(t)$, is considered as the IHCP. The solution is based on the knowledge of temperature values measured near the upper boundary. We consider two linear time varying functions of the strength of these heat sources:

$$g_1(t) = P_{11} + P_{12}\,t = 100 + 100\,t \qquad \text{and} \qquad g_2(t) = P_{21} + P_{22}\,t = 100 + 100\,t \tag{36}$$

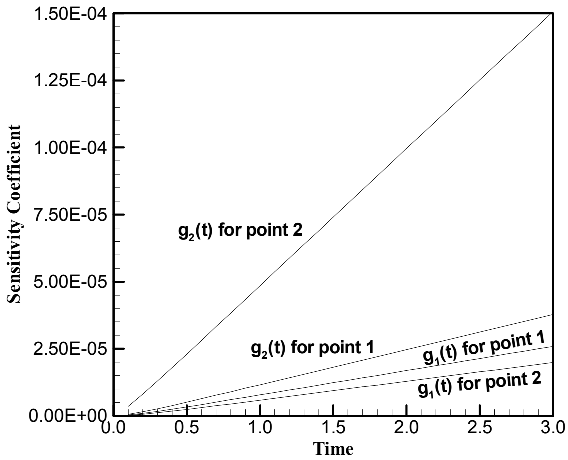
The two sources are located at $x_s = 0.3$, $y_s = 0.3$ and $x_s = 0.7$, $y_s = 0.7$. Also for solution of the inverse problem, two sensors are located at $x_s = 0.3, 0.7$, $y = 0.95$ for the Levenberg-Marquardt algorithm and four sensors are located at $x = 0.25, 0.3, 0.35, 0.7$, $y = 0.95$ for the adjoint conjugate gradient algorithm. Because source 1 is located far from sensor locations with respect to source 2, the number of the sensors required for estimation of source 1 in the adjoint conjugate gradient method is higher than those in the Levenberg- Marquardt



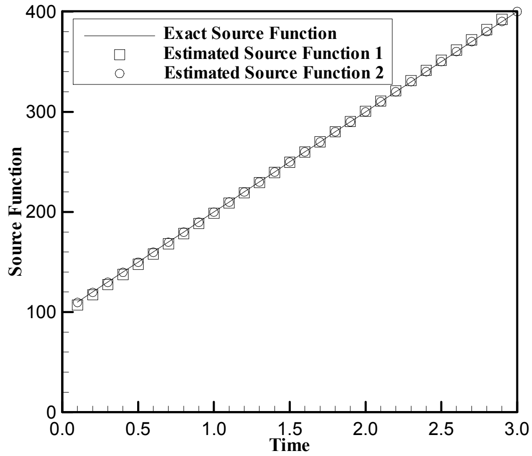
a. Time variation of sensitivity of the sensors to P_1 of function $g_1(t)$.



b. Time variation of sensitivity of the sensors to P_1 of function $g_2(t)$.



c. Time variation of sensitivity of the sensors to P_2 of heat source functions.



d. Variation of the heat source functions versus time.

Figure 5. Solution of case 2 using Levenberg-Marquardt algorithm..

Table 2: Simultaneous estimation of two time-varying heat sources in a two dimensional field.

Estimation Method	Function	Exact Measured Temperatures			Noisy Measured Temperatures		
		P_1	P_2	RMS* Error	P_1	P_2	RMS* Error
Levenberg-Marquardt	g_1	99.968	99.990	0.0492	99.863	101.84	1.8735
	g_2	100.00	99.998	0.0376	99.664	100.16	0.2885
Adjoint Conjugate Gradient	g_1	98.345	100.88	0.8983	97.383	100.33	2.0498
	g_2	100.16	99.947	0.1833	95.888	101.41	2.1347

*Root Mean Square

method. It is noticed that sensors that are located far from the source locations provide much less information because their sensitivity coefficients are small in comparison with sensors close to these locations. The solution of this case is presented in Figures 5 and 6 for the Levenberg-Marquardt and the adjoint conjugate gradient algorithms, respectively and the numerical results are presented in Table 2.

The timewise variations of the sensitivity of the noisy measured temperatures with respect to the parameters of functions $g_1(t)$ and $g_2(t)$ are shown in Figures 5.a, 5.b and 5.c. It can be seen that the sensitivities of the sensors with respect to parameter P_1 of these functions, increase continuously versus time, and then their values become constant. Also, the sensitivities of the sensors with respect to parameter P_2 of the functions go up linearly versus time. In Figure 5.d, the estimated functions have been compared with those of the exact functions for the noisy measured temperatures and show good agreement.

Figure 6.a shows the temperature field computed by the direct problem of the IHCP with two heat sources after estimating these functions, at $t = 3$. Also, in Figure 6.b, the estimated functions of the strengths of the heat sources have been compared with those of the exact functions for the noisy measured temperatures and show good agreement.

In second case, the strength functions of the heat sources have been selected as smooth linear functions and the positions of the heat sources with respect to each other are far enough so the inverse problem becomes nearly easy to solve. In fact, the problem could be more difficult, if source functions are too close to each other. In this case, simultaneous estimation of source functions or their coefficients is not trivial. In parameter estimation, it is not always obvious what parameters can be estimated. Sometimes it is only possible to estimate certain groups or the portion of parameters from the available measurements. Methods for determining what parameters/functions can be determined and what measurements are needed are available based on the study of the sensitivity coefficients, [24, 25, 28].

7. DISCUSSIONS

In this study, the transient two dimensional IHCPs are solved using multi block method and the Levenberg-Marquardt and the adjoint conjugate gradient algorithms. The inverse analyses are conducted globally in a two dimensional domain divided into two blocks with block-interface structured grids. The estimated strengths of the heat sources are compared with those of the exact functions. This study shows the ability of the multi block method along with inverse algorithms to estimate and to determine unknown parameters or functions. Also, this computer code can be used for estimation of unknown boundary conditions and for complex geometries.

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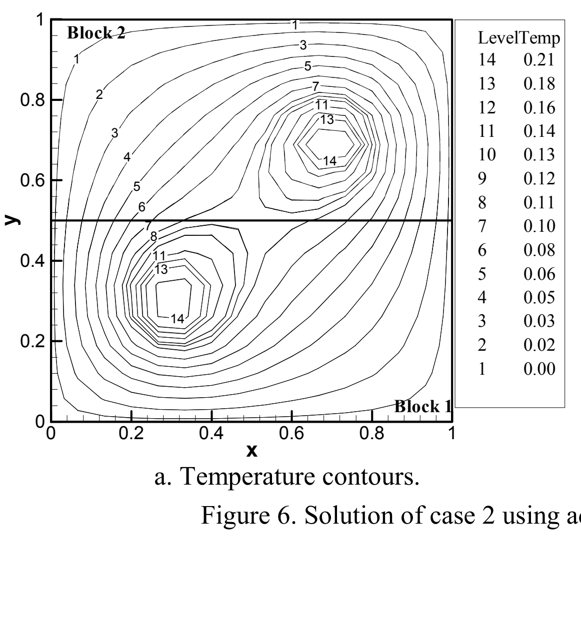
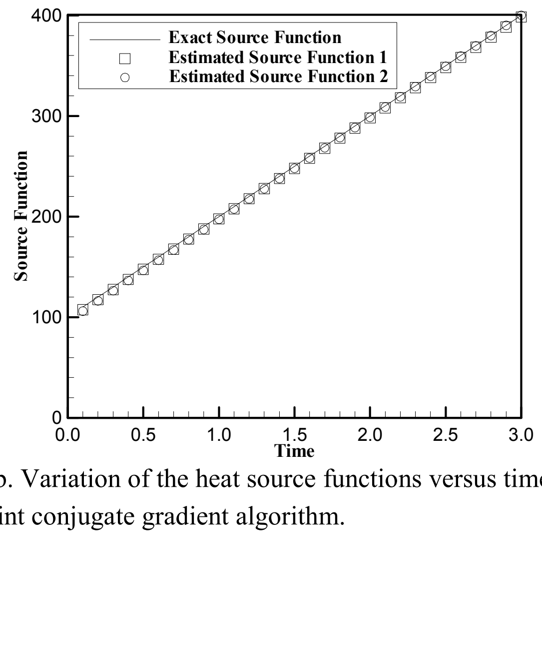
REFERENCES

1. J. Hadamard, *Lectures on: Cauchy's Problem in Linear Differential Equations*, Yale University Press, New Haven, CT, 1923.
2. J.V. Beck and K.A. Woodbury, Inverse problems and parameter estimation: Integration of measurements and analysis. *Meas. Sci. Technol.* (1998) **9**, 839-847.
3. J.V. Beck and K.J. Arnold, *Parameter Estimation in Engineering and Science*, John Wiley, New York, 1977.
4. A. Azimi, V. Esfahanian and S. Kazemzadeh Hannani, Simultaneous computations of internal-external axisymmetric laminar compressible flow using multi-block method, *Eleventh Annual Conference of the CFD (CFD03)*, CFD Society of Canada, Canada, 2003, pp. 437-444.
5. D.W. Pepper and J.C. Heinrich, *The Finite Element Method: Basic Concept and Applications*, Hemisphere, Washington, 1992.
6. A.J. Silva Neto and M.N. Ozisik, Two-dimensional inverse heat conduction problem of estimating the time-varying strength of a line heat source. *J. Appl. Phys.* (1992) **71**, 5357-5362.
7. A.J. Silva Neto and M.N. Ozisik, Inverse problem of simultaneously estimating the timewise-varying

strengths of two plane heat sources. *J. Appl. Phys.* (1993) **73** (5), 2132–2137.
8. C.H. Huang and M.N. Ozisik, Inverse problem of determining the unknown strength of an internal plane heat source. *J. Franklin Instit.* (1992) **329** (4), 751–764.
9. C. Le Niliot, D. Petit, Y. Touhami, and P. Gallet, An experimental estimation of a point heat source in a 2D diffusion system using different numerical techniques, *Second International Conference on Inverse Problems in Engineering: Theory and Practice*, Le Croisic, France, June, 1996, pp. 9-14.
10. C. Le Niliot, The boundary element method for the time varying strength estimation of point heat sources: Application to a 2D diffusion system. *Numer. Heat Transfer B* (1998) **33**, 301–321.
11. C. Le Niliot, F. Rigolleta and D. Petit, An experimental identification of line heat sources in a diffusive system using the boundary element method. *Int. J. Heat Mass Transfer* (2000) **43**, 2205-2220.
12. C. Le Niliot and F. Lefèvre, Multiple transient point heat sources identification in heat diffusion: Application to numerical 2D and 3D problems. *Numer. Heat Transfer B* (2001) **39**, 277–301.
13. C. Le Niliot and F. Lefèvre, A parameter estimation approach to solve the inverse problem of point heat sources identification. *Int. J. Heat Mass Transfer* (2004) **47**, 827–841.
14. J. Su and A.J. Silva Neto, heat source estimation with the conjugate gradient method in inverse linear diffusive problems. *J. Brazilian Soc. Mech. Sci.* (2001) **23** (3), 1-20.
15. F. Lefèvre and C. Le Niliot, Multiple transient point heat sources identification in heat diffusion: Application to experimental 2D problems. *Int. J. Heat Mass Transfer* (2002) **45** (9), 1951–1964.
16. C.Y. Yang, A sequential method to estimate the strength of the heat source based on symbolic computation. *Int. J. Heat Mass Transfer* (1998) **41** (14), 2245–2252.
17. C.Y. Yang, The determination of two heat sources in an inverse heat conduction problem. *Int. J. Heat Mass Transfer* (1999) **42**, 345-356.
18. G. Karami and M.R. Hematiyan, A boundary element method of inverse non-linear heat conduction analysis with point and line heat sources. *Commun. Numer. Meth. Eng*. (2000) **16**, 191–203.
19. R. Abou Khachfe and Y. Jarny, Determination of heat sources and heat transfer coefficient for 2-D heat flow: Numerical and experimental study. *Int. J. Heat Mass Transfer* (2001) **44** (7), 1309–1322.
20. E. Videcoq, D. Petit and A. Piteau, Experimental modeling and estimation of time varying thermal sources. *Int. J. Thermal Sci.* (2003) **42**, 255–265.
21. A. Azimi, S. Kazemzadeh Hannani and B. Farhanieh, Simultaneous estimation of heat source and boundary condition in two-dimensional transient inverse heat conduction problem, *Lecture Series on Computer and Computational Sciences: Int. Conf. of Computational Methods in Sciences and Engineering (ICCMSE 2004)*, Greece, 2004, pp. 49-52.
22. J.C. Tannehill, D.A. Anderson and R.H. Pletcher, *Computational Fluid Mechanics and Heat Transfer*, 2nd Edition, McGraw Hill, New York, 1997.
23. M.N. Ozisik and H.R.B. Orlande, *Inverse Heat Transfer: Fundamentals and Applications*, Taylor & Francis, New York, 2000.
24. K.J. Dowding, B.F. Blackwell and R.J. Cochran, Application of sensitivity coefficients for heat conduction problems, *7th AIAA/ASME Joint Therm. Heat Transfer Conf., June, 1998*.
25. K.J. Dowding, B.F. Blackwell and K.J. Cochran, Sensitivity computations for thermal problems, *5th US National Congress on Computational Mechanics (USNCCM99)*, August, 1999.
26. J.V. Beck, B. Blackwell and C.R. St Clair, *Inverse Heat Conduction*, John Wiley, New York, 1985.
27. K. Kurpisz and A.J. Nowak, *Inverse Thermal Problems*, Comp. Mech. Publications, Southampton, 1995.
28. A.F. Emery and A.V. Nenarokomov, Optimal experiment design. *Meas. Sci. Technol.* (1998) **9**, 864–76.



a. Temperature contours.



b. Variation of the heat source functions versus time.

Figure 6. Solution of case 2 using adjoint conjugate gradient algorithm.